A Study on Order Prime Signed Graphs of Finite Groups

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Contents

1 Introduction **2**

2 Objectives **2**

3 Outline of the Thesis **2**

3.1 CHAPTER 1: INTRODUCTION 3

3.1.1 Outline of the thesis 4

3.1.2 Basic definitions and preliminaries 4

3.2 CHAPTER 2: ORDER PRIME GRAPHS AND ARITHMETIC FUNCTION GRAPHS OF FINITE GROUPS 8

3.3 CHAPTER 3: SPECTRUM AND ENERGY OF ORDER PRIME GRAPHS AND ARITHMETIC FUNCTION GRAPHS 10

3.4 CHAPTER 4: ORDER PRIME SIGNED GRAPHS AND ARITHMETIC FUNCTION SIGNED GRAPHS 13

3.5 CHAPTER 5: SPECTRUM ENERGY OF ORDER PRIME SIGNED GRAPHS AND ARITHMETIC FUNCTION SIGNED GRAPHS 16

3.6 CHAPTER 6: CONCLUSION AND SCOPE FOR FUTURE WORK . 18

4 List of Publications/Papers Presented/ Communicated **24**

4.1 Publications in Journals/ Communicated 24

4.2 Papers presented in an International/National conference 24

4.3 Workshops/Conference attended 25

1 Introduction

For over a century, algebraic structures and graphs have been closely related. Many studies have used the strong combinatorial characteristics of graphs to examine the algebraic characteristics of rings and groups, and vice versa. This work is obviously no exception. Researchers appear to be very interested in and active in the area of graph-ring associating research lately. The volume of publications adopting this viewpoint makes this evident, see [4, 7, 8, 58, 91, 92, 96, 97]. The unit elements of the commutative ring R have been utilized to attribute graphs with non-zero identity, [8]. Graphs on rings have been described by some researchers using zero-divisors in a ring R, see [4, 7, 58], while others have utilized the co-maximal ideals of both commutative and non-commutative rings, see [91, 92].

Graph theory is extremely important and has many applications outside of mathematics. Its importance cannot be overstated. It is extensively used in many disciplines, including Computer Science, Biology, Electrical Engineering, Biochemistry, Sociology, Chemical engineering, Operation research, Landscape Ecology, and Geosciences, etc. Graph combinatorial ideas have been applied in group theory to resolve some eigenstate issues pertaining to "Lie groups", see [62]. Associated graphs have also made it simple to verify a number of other group theory results, [16].

2 Objectives

In this thesis we have planned the following major objectives.

1. To study order prime graphs through arithmetic function graphs of finite groups
2. To study spectrum and energy of order prime graphs through arithmetic function graphs
3. To study order prime signed graphs through arithmetic function signed graphs
4. To study spectrum energy of order prime signed graphs through arithmetic function signed graphs.

3 Outline of the Thesis

The thesis "A Study on Order Prime Signed Graphs of Finite Groups" is divided into six chapters.

Chapter 1 : INTRODUCTION

Chapter 2 : ORDER PRIME GRAPHS AND ARITHMETIC FUNCTION GRAPHS OF FINITE GROUPS

Chapter 3 : SPECTRUM AND ENERGY OF ORDER PRIME GRAPHS AND ARITHMETIC FUNCTION GRAPHS

Chapter 4 : ORDER PRIME SIGNED GRAPHS AND ARITHMETIC FUNCTION SIGNED GRAPHS

Chapter 5 : SPECTRUM AND ENERGY OF ORDER PRIME SIGNED GRAPHS AND ARITHMETIC FUNCTION SIGNED GRAPHS

Chapter 6 : CONCLUSION AND SCOPE FOR FUTURE WORK

3.1 CHAPTER 1: INTRODUCTION

A graph is the simplest mathematical model of a system described by a set of elements and a relation that may or may not exist between any two elements. A graph can therefore represent the simplest aspect of a communication network, a sociogram or a nervous system. In particular it can also represent a contagion process, where the points/vertices of the graphs represent individuals and the edges/links represent contacts between pairs of individuals.

A system represented by a graph can be thought of as a result of a stochastic process. For instance, starting with a population of points as given, we can imagine that links start growing from points ultimately to connect with other points. The stochastic process would be described by the probabilistic rules governing the selection of the points of origin or the "targets" of the links, or, perhaps of pairs of points to be connected by links. A graph is a particular realization resulting from applying such probabilistic rules. Repeated application of the rules produces a population of graphs. One of the main issues is figuring out the likelihood that a randomly chosen graph from the population, created by repeatedly applying the probabilistic rules, would have a particular feature of a graph. Alternatively, we may look at the population of all feasible graphs of a certain kind and inquire as to what the likelihood is that a graph chosen at random from that population will possess a particular attribute. Clearly in this latter case, assuming that each graph has an equal probability of being selected, the probability sought will be the ratio of the number of graphs having the property in question to the total number of graphs.

Group theorists have approached the subject of finite groups in a wide range of ways over the years. From computational studies in a specific individual group

to combinatorial arguments over the characteristics of families of abstract groups. There are numerous study avenues available in this content-rich topic. Analyzing combinatorial structures that a group acts upon is one such approach. This method not only discloses group-related information, but the structures themselves are frequently fascinating on their own. This method has been successfully applied in several research over the past 40 years when the structure in question is a graph or an incidence geometry.

3.1.1 Outline of the thesis

Chapter 1 is an introductory chapter that contains history and motivation for the study. It is intended to introduce basic definitions, terminologies and concepts related to our study.

The order prime graph of a finite group and its generalizations are covered in Chapter 2. We define the arithmetic function graph of a finite group with respect to an arithmetical function as a generalization of an order prime graph. It is noted that the arithmetic function graph with respect to the Euler's ϕ-function is all that a finite group's order prime graph is. We study certain results about the isomorphism, planarity, diameter, and dominating sets of arithmetic function graphs of finite groups.

There are many important concepts related to chemistry involving groups and graphs based on Spectrum and energy of graphs. Spectrum and energy are important tools that help to solve the structural identification problem in chemical graph theory. This motivated us to study eigenvalues and energy of arithmetic function graphs. In Chapter 3, we discuss some results on eigenvalues and energy of Arithmetic function graphs of finite groups.

The order prime signed graph of a finite group and its generalizations are covered in Chapter 4. We provide two more extensions of the order prime signed graph: the set-prime signed graph and the arithmetic function signed graph with regard to an arithmetical function. Additionally, we showcase certain findings concerning these two signed graphs associated to a finite group.

We revisit the ideas of spectrum and energy of signed graphs in Chapter 5. We show some results and analyze the spectrum and energy of signed graphs of order prime signed graphs and arithmetic function signed graphs of finite groups.

Finally, we covered the conclusions and areas for future research in Chapter 6.

3.1.2 Basic definitions and preliminaries

Definition 3.1. *A Groupoid is a non-empty set together with a binary operation.*
Associative groupoid is referred to as a semigroup.
A semigroup with identity element is known as a monoid.
A group is nothing but a monoid, in which every element has a unique inverse.
A group $\mathfrak{A}$ *is abelian if and only if* $uv = vu$, $\forall\ u, v \in \mathfrak{A}$.

An infinite group is one that does not have a finite number of elements; a finite group does.

Definition 3.2. *A subset M of a group $\mathfrak{A}$ is called a generating set of $\mathfrak{A}$ if every element of $\mathfrak{A}$ can be expressed as a product (under the group operation) of finitely many elements of M and their inverses.*

Given a group $\mathfrak{A}$ and a subset M, the smallest subgroup of $\mathfrak{A}$ that contains all of the elements of M is the subgroup generated by M, denoted by $\langle M \rangle$. $\langle M \rangle$ is the intersection of all subgroups of $\mathfrak{A}$ that contains all of the elements of M. The elements in M are referred to as generators or group generators if $\mathfrak{A} = \langle M \rangle$. In this case, we say M generates $\mathfrak{A}$.

A group $\mathfrak{A}$ is a cyclic group if there is an element $u \in \mathfrak{A}$ such that $\langle u \rangle = \mathfrak{A}$.

If all the elements in a group $\mathfrak{A}$ are of the form u^k for some integer k, then $\mathfrak{A}$ is cyclic with generator u. We write ku instead of u^k if the operation is addition.

Definition 3.3. *The number of elements in a group $\mathfrak{A}$ is the order of $\mathfrak{A}$. It is expressed as $|\mathfrak{A}|$ or $o(\mathfrak{A})$.*

Definition 3.4. *The smallest positive integer m such that $u^m = e$, where e is the group identity, is reffered to as the order of an element u in a group $\mathfrak{A}$. In the event that m does not exist, the element u has infinite order.*

The order of an element u in a group $\mathfrak{A}$ is denoted by $|u|$ or $o(u)$.

Unless otherwise indicated, $\mathfrak{A}$ will represent a finite group in this thesis.

A graph is defined as a finite nonempty set V of p vertices and a collection E of q unordered pairs, or edges, between different vertices of V. Evidently $0 \leq q \leq \binom{p}{2}$. Vertices u and v are said to be joined by the edge $x = \{u, v\}$. When two vertices, u and v, are adjacent, and an edge, x, is incident with both, we say that they are so. The complete graph K_p has every pair of verticess adjacent and $q = \binom{p}{2}$ while the totally disconnected graph $\overline{K_p}$ has no adjacencies, so that $q = 0$. The trivial graph has one vertex and no edges; it is both complete and totally disconnected. A rooted graph has one vertex, the root, distinguished. A multigraph is defined similarly, but can have more than one edge joining a pair of verticess. A directed graph, also known as a digraph, is made up of a vertex set V and a collection X of ordered pairs, or directed edges, that are q distinct vertices. As we can see, $0 \leq q \leq p(p-1)$. We say that u is adjacent to v and v is adjacent from u if the ordered pair (u, v) is in E.

The degree of a vertex u is the number of vertices that are adjacent to it. If a vertex u is not adjacent to another vertex in G, then it is considered isolated. We refer to u as a leaf if it shares a boundary with just one vertex. An r-regular graph is a graph G with vertices of degree r each. Parallel edges are two or more edges that connect the same pair of different vertices. An edge that connects a vertex to itself is called a loop. If a graph has no loops or parallel edges, it is considered simple. In this thesis, we will just look at simple graphs, which we will just refer to as graphs.

When it comes to graphs G, the highest degree among the vertices of $V(G)$ is called the maximum degree and is represented by $\Delta(G)$, while the lowest degree is represented by $\delta(G)$.

An alternating sequence of unique vertices and edges $v_0, e_1, v_1, \cdots, e_n, v_n$, starting and ending with vertices, is called a graph's path. Each edge is incident with the vertices that come before and after it. A sequence is a cycle if $v_0 = v_n$. A tree is a connected graph for which $p = q + 1$. A graph is considered connected if every pair of vertices is related by a path.

If there is a mapping that preserves adjacency between the vertices of G_1 and G_2, then two graphs, $G_1 \cong G_2$, are isomorphic; this mapping is known as an isomorphism. An automorphism is an isomorphism of a graph onto itself. All automorphisms of a graph G make up its group, $\mathfrak{G}(G)$; it is a permutation group acting on the set V of vertices.

If $v_i v_j \in E(G)$, then $a_{ij} = 1$; otherwise, $a_{ij} = 0$. Given a graph G with $V(G) = \{v_1, v_2, \cdots, v_n\}$, the adjacency matrix of G, denoted by $A(G)$, is the $n \times n$, symmetric, $(0, 1)$ matrix (a_{ij}). Since $A(G)$ is symmetric and each main diagonal entry is 0, all information describing the graph is contained in those entries which lie above the main diagonal. We shall sometimes focus our attention on this array of entries only, rather than on the whole of $A(G)$.

From the definitions of all these matrix graphs we find that the actual values of the non-zero elements of the matrix do not play any part in these definitions. So all the non-zero elements of the matrix can be replaced by l's and thus we get a $(0, 1)$ matrix from an arbitrary matrix and all the graphs defined above can be said to be graphs associated with a $(0, 1)$ matrix obtained from an arbitrary matrix.

By labelling vertices, edges, cliques, cycles or other subgraphs of a graph we can associate several $(0, 1)$ matrices with the graph and all these matrices give some information about the graph. Some of these matrices characterise the graph completely and some of them only partially. There are a few non-$(0, l)$ matrices which can be associated with a graph, namely distance matrix, degree matrix etc. It is found that $(0, 1)$ matrices of graphs and graphs of $(0, 1)$ matrices can be used to get information about each other.

The eigenvalues of an adjacency matrix of a graph are the eigenvalues (λ such that $Ax = \lambda x$ for $x \neq 0$). The eigenvalues are the graph's numerical invariants in graph theory. Regretfully, the graph cannot be uniquely identified by the eigenvalues. It was demonstrated by Collatz and Sinogowitz [25] that graphs that are not isomorphic can nonetheless have the same eigenvalues.

The characteristic polynomial of the graph G is the determinant $\det(A - \lambda I)$. It is possible to define the eigenvalues of G as the characteristic polynomial's roots. The dimension of an eigenspace corresponding to an eigenvalue A is its geometric multiplicity. The multiplicity of A as a root of the characteristic polynomial is the algebraic multiplicity of an eigenvalue A. The geometric multiplicity of an eigenvalue

is usually equal to or larger than its algebraic multiplicity.

In Chemistry, a molecule can he drawn as a graph, with the atoms as vertices, and bonds as edges. The first investigation of the eigenvalues of a graph was by two chemists, Coulson and Rushbrooke [27]. They studied the Huckel approximation to Schrodinger's equation. For a class of hydrocarbons known as conjugated polyens, if 0 is an eigenvalue of the graph of a molecule, then the molecule cannot have total electron spin equal to zero. This implies that the molecule is chemically unstable.

In 1957, Collatz and Sinogowitz [25] initiated the study of the spectrum of a graph. Their pioneering paper answered negatively the first question that seems to occur to each new investigator in this field: "Does the adjacency matrix's characteristic polynomial define the graph?" Actually, the data compiled for trees in [25] already contains as a counterexample two cospectral trees with 8 points.

The set of pairings

$$S = \{(\lambda_0, m(\lambda_0), (\lambda_1, m(\lambda_1), \cdots, (\lambda_v, m(\lambda_v))\}$$

on the graph G is the spectrum, where λ_i, $i = 0, 1, \cdots, v$ are the distinct eigenvalues of the matrix A and $m(\lambda_i)$, represents their multiplicities.

The radius of the smallest disk containing the whole spectrum that is centered at the origin is called the spectral radius, or $\rho(G)$. Thus $\rho(G) = \max|\lambda_i|$.

Fisher [34] gathered evidence to suggest the correct answer to a question of Mark Kac: "Can the shape of a drum be heard?" The mathematical meaning of this vividly worded riddle is that if one is given all the eigenvalues for modes of oscillation of a 2-dimensional drum head, is this enough information to determine the shape of the drum? The answer is almost certainly no, yet no counterexample for Kac's question has yet been found. Generalizations to higher dimensions have produced counterexamples, but none are known in fewer than 16 dimensions! The graphical question mentioned above is a finite discrete analog of Kac's question, which motivated Fisher's interest. In addition, Fisher reported that even if one cannot "hear the shape of a drum", still significant specific information can be obtained, such as the area and perimeter of the drum. Similarly, several graphical properties are found to be spectrally determined.

Finally, for the 3-dimensional version of Kac's question, it has been found that at least solids of revolution can be "heard". This suggests that one should attempt to decide whether certain special families of graphs are spectrally determined.

M. Sattanathan and R. Kala [80] proposed the idea of an order prime graph. Additionally, order prime graphs of finite groups were examined by Ma et al. [59] and Dorbidi [32], although they were referred to as coprime graphs. You may find the following definition in [80]: Any two different vertices a and b are adjacent in $OP(\mathfrak{A})$ if and only if $(o(a), o(b)) = 1$. This is the order prime graph $OP(\mathfrak{A})$ of a finite group $\mathfrak{A}$.

R. Rajendra and P. S.K. Reddy [68, 69] proposed the idea of a general order prime graph. A graph with vertex set $V(GOP(\mathfrak{A})) = \mathfrak{A}$ and any two different vertices a and b are adjacent in $GOP(\mathfrak{G})$ if and only if $(o(a), o(b)) = 1$ or p, where p is a prime and $p < n$. This is known as the general order prime graph $GOP(\mathfrak{A})$ of $\mathfrak{A}$. It is obvious that a subgraph of $GOP(\mathfrak{A})$ is $OP(\mathfrak{A})$.

The authors of [72] looked into a few characteristics of prime graphs of finite rings and developed a method that uses the gcd-sum function to find the number of edges in the prime graph of the ring of residue classes modulo n. Further, discussed some results regarding eigenvalues and energy of prime graphs of finite rings.

Let S be a positive integer set that is not empty. As a graph with vertex set $V(G_S(\mathfrak{A})) = \mathfrak{A}$ and any two vertices a and b are adjacent in $G_S(\mathfrak{A})$ if and only if $(o(a), o(b)) \in S$, the set-prime graph $G_S(\mathfrak{A})$ of a given finite group $\mathfrak{A}$ of order n with regard to S [71]. Order prime and general order prime graphs are special examples of set-prime graphs, as noted by Rajendra et al. [71], who also looked at several aspects of set-prime graphs of finite groups.

3.2 CHAPTER 2: ORDER PRIME GRAPHS AND ARITHMETIC FUNCTION GRAPHS OF FINITE GROUPS

Throughout this chapter, $\mathfrak{A}$ denotes a finite group and we denote the identity element of $\mathfrak{A}$ by e. The order of an element a in a group $\mathfrak{A}$ is denoted by $|a|$ or $o(a)$; and the order of $\mathfrak{A}$ is denoted by $|\mathfrak{A}|$ or $o(\mathfrak{A})$. The group of residue classes modulo n is denoted by $\mathbb{Z}_n$. The Klein's 4-group is denoted by V_4.

Let $\mathfrak{A}$ be a finite group of order n and let S be a non-empty set of positive integers. We recall the definitions of order prime graph and its generalizations.

Definition 3.5 (Order prime graph [80])**.** *The order prime graph $OP(\mathfrak{A})$ of $\mathfrak{A}$ is defined as a graph with the vertex set $V(OP(\mathfrak{A})) = \mathfrak{A}$ and any two vertices a and b are adjacent in $OP(\mathfrak{A})$ if and only if* $\gcd(|a|, |b|) = 1$.

Definition 3.6 (General order prime graph [68, 69])**.** *The general order prime graph $GOP(\mathfrak{A})$ is defined as a graph with the vertex set $V(GOP(\mathfrak{A})) = \mathfrak{A}$ and any two vertices a and b are adjacent in $GOP(\mathfrak{A})$ if and only if* $\gcd(|a|, |b|) = 1$ *or p, where p is a prime and $p < n$.*

It is obvious that the graph $OP(\mathfrak{A})$ is a subgraph of the graph $GOP(\mathfrak{A})$.

Definition 3.7 (Set-prime graph [71])**.** *The set-prime graph $G_S(\mathfrak{A})$ of $\mathfrak{A}$ with respect to a set $S \subseteq \mathbb{N}$ is defined as a graph with vertex set $V(G_S(\mathfrak{A})) = \mathfrak{A}$ and any two vertices a and b are adjacent in $G_S(\mathfrak{A})$ if and only if* $\gcd(|a|, |b|) \in S$.

Definition 3.8 (Prime coprime graph [3])**.** *The prime coprime graph $\Theta(\mathfrak{A})$ of a finite group $\mathfrak{A}$ is defined as a graph with the vertex set $V(\Theta(\mathfrak{A})) = \mathfrak{A}$ and any two vertices x, y in $\Theta(\mathfrak{A})$ are adjacent if and only if* $\gcd(|x|, |y|)$ *is equal to* 1 *or a prime number.*

The general order prime graph and the prime coprime graph coincide for all finite groups except for the cyclic groups of prime order. Also, the prime coprime graph of a

finite group which coincides with the concepts of the set-prime graph $G_S(\mathfrak{A})$ of a finite group $\mathfrak{A}$ for some particular set S (see [73]). Another generalization of the concept of order prime graph involving arithmetic functions is defined in this chapter.

Definition 3.9. *The Arithmetic function graph $G_h(\mathfrak{A})$ of $\mathfrak{A}$ with respect to an arithmetical function h is defined as a graph with vertex set $V(G_h(\mathfrak{A})) = \mathfrak{A}$ and two vertices a and b are adjacent in $G_h(\mathfrak{A})$ if and only if $h(|a||b|) = h(|a|)h(|b|)$.*

Rani Jose and D. Susha [54] have studied the μ-graph of a finite group which is nothing but an arithmetic function graph with $h = \mu$. In this section, it is shown that the order prime graph $OP(\mathfrak{A})$ is nothing but the arithmetic function graph $G_\phi(\mathfrak{A})$ with respect to the Euler's ϕ-function (see Corollary **??**).

Here, we mentioned some important results under this chapter:

Proposition 3.10. *Let h be an arithmetical function. If $\mathfrak{A}_1$ is a subgroup of a finite group $\mathfrak{A}_2$, then $G_h(\mathfrak{A}_1)$ is a subgraph of $G_h(\mathfrak{A}_2)$.*

Proposition 3.11. *Let $\mathfrak{A}$ be a finite group. If h is a completely multiplicative function, then $G_h(\mathfrak{A})$ is a complete graph.*

Proposition 3.12. *Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be two finite groups and let h be a multiplicative function. If $\mathfrak{A}_1 \cong \mathfrak{A}_2$, then $G_h(\mathfrak{A}_1) \cong G_h(\mathfrak{A}_2)$.*

Proposition 3.13. *Let $\mathfrak{A}$ be a finite group of order n. If h is an arithmetical function such that $h(1) = 1$, then*

(1) $G_h(\mathfrak{A})$ is a connected graph,

(2) $\{e\}$ is a dominating set,

(3) $\deg(e) = n - 1$ and $\Delta(G_h(\mathfrak{A})) = n - 1$,

(4) $diam(G_S(\mathfrak{A})) \leq 2$.

(5) $(n-1) \leq |E(G_h(\mathfrak{A}))| \leq \binom{n}{2}$.

Theorem 3.14. *Let $\mathfrak{A}$ be a finite group. If h is a multiplicative function, then $OP(\mathfrak{A})$ is a subgroup of $G_h(\mathfrak{A})$.*

Theorem 3.15. *Let $\mathfrak{A}$ be a finite group of order $p > 2$, where p is a prime, and h be a multiplicative function. Then*

(i) $G_h(\mathfrak{A})$ is a star if and only if $h(p^2) \neq h(p)^2$,

(ii) $G_h(\mathfrak{A})$ is a complete graph if and only if $h(p^2) = h(p)^2$.

Theorem 3.16. *Let $\mathfrak{A}$ be a finite group of order n and h be a multiplicative function. If n is a composite number that is not a power of a prime, then $G_h(\mathfrak{A})$ is not a star.*

Theorem 3.17. *Let h be a multiplicative function such that $h(r^2) \neq h(r)^2$ for any prime r. If $\mathfrak{A}$ is a non-abelian group of order pq, where p and q are distinct primes with $p < q$, then*

(i) $G_h(\mathfrak{A})$ *is a complete tripartite graph isomorphic to* $K_{1,r,s}$*, where* r *and* s *are the number of elements of order* p *and* q*, respectively;*

(ii) $\chi(G_h(\mathfrak{A})) = 3$*;*

(iii) $G_h(\mathfrak{A})$ *is non-planar of diameter* 2*.*

3.3 CHAPTER 3: SPECTRUM AND ENERGY OF ORDER PRIME GRAPHS AND ARITHMETIC FUNCTION GRAPHS

The letter $\mathfrak{A}$ represents a finite group and n denotes a positive integer, throughout this chapter. The order of an element a in a group $\mathfrak{A}$ is denoted by $|a|$ or $o(a)$; and the order of $\mathfrak{A}$ is denoted by $|\mathfrak{A}|$ or $o(\mathfrak{A})$. Given two numbers, x and y, their greatest common divisor (gcd) is denoted by (x, y) or $\gcd(x, y)$.

We recall the following basic definitions and results:
Consider a graph G with n vertices, such as $v_1, v_2, \ldots, v_n$. The square matrix of order n that represents the adjacency matrix of G is $A = A(G)$. Its (i, j)-entry is defined as:

$$A_{ij} = \begin{cases} 1, & \text{if there is an edge } e \text{ joined by } v_i \text{ and } v_j \\ 0, & \text{otherwise.} \end{cases}$$

We refer to the eigenvalues of G as the eigenvalues of $A(G)$. We refer to the set of eigenvalues of G as spectrum G, and we represent it with the notation $Spec(G)$. The spectrum of G is as follows if the different eigenvalues of G are $\lambda_1 > \lambda_2 > \cdots > \lambda_k$ and their multiplicities are $m_1, m_2, \ldots, m_k$:

$$Spec(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ m_1 & m_2 & \cdots & m_k \end{pmatrix} \begin{matrix} \leftarrow \text{eigenvalues} \\ \leftarrow \text{multiplicity} \end{matrix}$$

We denote largest and smallest eigenvalues of a graph G by λ_{max} and λ_{min} respectively. When a graph G has a subgraph H, then we have

$$\lambda_{max}(G) \geq \lambda_{max}(H). \tag{3.1}$$

For any graph G,

$$\lambda_{max} \leq d_{max}, \tag{3.2}$$

in which d_{max} denotes the graph G's maximum vertex degree (See [11]).

Given a graph G, let the eigenvalues be $\lambda_1, \lambda_2, \ldots, \lambda_n$. According to [11, 38, 39, 40], the energy of G is defined as:

$$\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i|. \tag{3.3}$$

For $K_{1,n-1}$, the spectrum is:

$$Spec(K_{1,n-1}) = \begin{pmatrix} \sqrt{n-1} & 0 & -\sqrt{n-1} \\ 1 & n-2 & 1 \end{pmatrix}. \tag{3.4}$$

For $K_{1,n-1}$, the energy is:

$$\mathcal{E}(K_{1,n-1}) = 2\sqrt{n-1}. \tag{3.5}$$

For K_n, the spectrum is:

$$Spec(K_n) = \begin{pmatrix} n-1 & -1 \\ 1 & n-1 \end{pmatrix}. \tag{3.6}$$

For K_n, the energy is:

$$\mathcal{E}(K_n) = 2n-2. \tag{3.7}$$

The following are the some important results on Chapter 3:

Theorem 3.18. *Consider a group $\mathfrak{A}$ of finite order. Then*

(i) $o(\mathfrak{A}) = p^\alpha$ where p is a prime and $\alpha \in \mathbb{Z}^+$ if and only if for each eigenvalue λ of $OP(\mathfrak{A})$, $-\lambda$ is an eigenvalue with the same multiplicity.

(ii) $o(\mathfrak{A}) = p^\alpha$ where p is a prime and $\alpha \in \mathbb{Z}^+$ if and only if

$$\lambda_{min}(OP(\mathfrak{A})) = -\lambda_{max}(OP(\mathfrak{A})).$$

Theorem 3.19. *Given a finite group $\mathfrak{A}$ of order $n \geq 3$, $OP(\mathfrak{A})$ has atleast three distinct eignvalues.*

Theorem 3.20. *Let $\mathfrak{A}$ be a finite group.*

(i) If $o(\mathfrak{A}) = 2$, then $OP(\mathfrak{A}) \cong K_2$ and $\mathcal{OPE}(\mathfrak{A}) = 2$.

(ii) If $o(\mathfrak{A}) = n$, then

$$\mathcal{OPE}(\mathfrak{A}) \leq \frac{n(\sqrt{n}+1)}{2}. \tag{3.8}$$

(iii) If $o(\mathfrak{A}) = n = p^\alpha$ where p is a prime and $\alpha \in \mathbb{Z}^+$, then

$$\mathcal{OPE}(\mathfrak{A}) \leq \frac{n(\sqrt{n}+\sqrt{2})}{\sqrt{8}} \tag{3.9}$$

(iv) If $o(\mathfrak{A}) = n = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$ where p_i's are primes and $n_i \in \mathbb{Z}^+, \forall i$, then

$$\mathcal{OPE}(\mathfrak{A}) \leq \sqrt{n\left(n^2 - n + \sum_{i=1}^{k} x_i - \sum_{i=1}^{k} x_i^2\right)} \tag{3.10}$$

where x_i is the number of elements in $\mathfrak{A}$ of order p_i^j, $1 \leq j \leq n_i$, $1 \leq i \leq k$.

The energy and spectrum of arithmetic function graphs of finite groups are covered in this section. The h-energy of a group is defined with regard to an h be an arithmetic function, and certain results are obtained.

Definition 3.21. *Let h be an arithmetic function and $\mathfrak{A}$ be a finite group. The energy of the arithmetic function graph $G_h(\mathfrak{A})$ is an expression of the h-energy of $\mathfrak{A}$, represented by $\mathcal{E}_h(\mathfrak{A})$. In other words, $\mathcal{E}_h(\mathfrak{A}) = \mathcal{E}(G_h(\mathfrak{A}))$.*

Theorem 3.22. *If h is a multiplicative function and $\mathfrak{A}$ is a group with $o(\mathfrak{A}) = n$, then*

$$max\left\{\frac{2n-2}{n},\ \sqrt{n-1}\right\} \leq \lambda_{max}(G_\phi(\mathfrak{A})) \leq \lambda_{max}(G_h(\mathfrak{A})) \leq n-1. \tag{3.11}$$

Especially, if $n \geq 3$, $\sqrt{n-1} \leq \lambda_{max}(G_\phi(\mathfrak{A})) \leq \lambda_{max}(G_h(\mathfrak{A})) \leq n-1$.

Theorem 3.23. *With p being a prime number and h being a multiplicative function, let $\mathfrak{A}$ be a finite group of rank $p > 2$. Then*

(i) $h(p^2) \neq h(p)^2$ *if and only if for each eigenvalue λ of $G_h(\mathfrak{A})$, $-\lambda$ is an eigenvalue with the same multiplicity.*

(ii) $h(p^2) \neq h(p)^2$ *if and only if*

$$\lambda_{min}(G_h(\mathfrak{A})) = -\lambda_{max}(G_h(\mathfrak{A})).$$

Theorem 3.24 ([73, Theorem 3])**.** *$G_\phi(\mathfrak{A})$ has at least three distinct eignvalues if $\mathfrak{A}$ is a finite group of order $n \geq 3$.*

Theorem 3.25. *For any prime r, let h be a multiplicative function such that $h(r^2) \neq h(r)^2$. At least three distinct eignvalues exist for $G_h(\mathfrak{A})$ if $\mathfrak{A}$ is a non-abelian group of order pq, where p and q are distinct primes with $p < q$.*

Theorem 3.26. *[73, Theorem 5] Consider the finite group $\mathfrak{A}$.*

(i) When $o(\mathfrak{A}) = 2$, subsequent to $G_\phi(\mathfrak{A}) \cong K_2$ alongside $\mathcal{E}_\phi(\mathfrak{A}) = 2$.

(ii) If $o(\mathfrak{A}) = n$, then

$$\mathcal{E}_\phi(\mathfrak{A}) \leq \frac{n(\sqrt{n}+1)}{2} \tag{3.12}$$

(iii) Given that $\alpha \in \mathbb{Z}^+$ and that $o(\mathfrak{A}) = n = p^\alpha$ for all primes p,

$$\mathcal{E}_\phi(\mathfrak{A}) \leq \frac{n(\sqrt{n}+\sqrt{2})}{\sqrt{8}} \tag{3.13}$$

(iv) If $o(\mathfrak{A}) = n = p_1^{n_1}p_2^{n_2}\cdots p_k^{n_k}$ where p_i's are primes and $n_i \in \mathbb{Z}^+, \forall i$, then

$$\mathcal{E}_\phi(\mathfrak{A}) \leq \sqrt{n\left(n^2 - n + \sum_{i=1}^{k} x_i - \sum_{i=1}^{k} x_i^2\right)} \tag{3.14}$$

where x_i is the number of elements in $\mathfrak{A}$ of order p_i^j, $1 \leq j \leq n_i$ and $1 \leq i \leq k$.

Theorem 3.27. *With q being a prime number and h being a multiplicative function, for a finite group $\mathfrak{A}$ of order $q > 2$, we have the following:*

(i) If $h(q^2) \neq h(q)^2$, then $G_h(\mathfrak{A})$ is a graph with a spectrum

$$Spec(G_h(\mathfrak{A})) = \begin{pmatrix} \sqrt{q-1} & 0 & -\sqrt{q-1} \\ 1 & q-2 & 1 \end{pmatrix}$$

and

$$\mathcal{E}_h(\mathfrak{A}) = 2\sqrt{q-1}.$$

(ii) The graph $G_h(\mathfrak{A})$ has a spectrum if $h(q^2) = h(q)^2$:

$$Spec(G_h(\mathfrak{A})) = \begin{pmatrix} q-1 & -1 \\ 1 & q-1 \end{pmatrix}$$

and

$$\mathcal{E}_h(\mathfrak{A}) = 2(q-1).$$

3.4 CHAPTER 4: ORDER PRIME SIGNED GRAPHS AND ARITHMETIC FUNCTION SIGNED GRAPHS

A signed graph can be created by adding a sign to each edge of a simple graph. In 1953, Harary presented the idea of signed graphs [44] in relation to a few social psychology issues. In 1982, Zaslavsky introduced matroids of signed graphs [98]. Signed graphs are very important as they are used in the study of complex systems of computer models and sociology. R. Rajendra et al. [70] defined two signed graphs associated with a finite group, namely order prime signed graph and general order prime signed graph, taking into consideration the commuting property of elements in finite non-abelian groups.

The order prime signed graph of a finite group and its generalizations are discussed in this chapter. We give two more generalizations of order prime signed graph, namely, set-prime signed graph and arithmetic function signed graph w.r.to an arithmetical function and we present some results involving these two signed graphs associated with a finite group.

An ordered pair $\mathcal{S} = (G, \sigma)$ is referred to as a signed graph [44, 98] when $G = (V, E)$ is a graph known as the underlying graph of $\mathcal{S}$ and $\sigma : E \to \{+, -\}$ is a function.

If there are an even number of negative edges in each cycle of a signed graph $\mathcal{S} = (G, \sigma)$, then $\mathcal{S}$ is balanced [44]. In contrast, if the product of the signs of the edges on each cycle of $\mathcal{S}$ is positive, then a signed graph is said to be balanced.

If each and every edge in a signed graph has the same sign, the graph is said to be homogenous; if not, it is said to be heterogeneous. The signed graph that results from reversing the signs on the edges of a signed graph $\mathcal{S}$ is called its negation of $\mathcal{S}$ and is denoted by $\neg\mathcal{S}$.

A graph can be thought of as an all-positive signed graph, and signed graphs are a generalization of graphs in this sense. A signed graph's sign is equal to the product of the signs of its edges.

If the underlying undirected graphs of two signed graphs are isomorphic and the signs on the edges are retained, then the two signed graphs are isomorphic. We assume that $\mathfrak{A}$ is a finite group with order n. We revisit the definitions of $\mathfrak{A}$'s order prime graph and its generalization, general order prime signed graph. An edge with end vertices u and v in a graph is denoted by uv or $\overline{(u, v)}$.

Definition 3.28 (Order prime signed graph [70])**.** *The order prime signed graph $OPS(\mathfrak{A})$ of a finite group $\mathfrak{A}$ is the signed graph $((OP(\mathfrak{A}), \sigma)$ where the function $\sigma : E(OP(\mathfrak{A})) \to \{+, -\}$ is given by*

$$\sigma(\overline{(a, b)}) = \begin{cases} +, & \text{if } ab = ba; \\ -, & \text{otherwise.} \end{cases}$$

It can be shown from the Definition 3.28 that if a group $\mathfrak{A}$ is abelian, then all of $OPS(\mathfrak{A})$'s edges are of the + sign, thus in this instance, $OPS(\mathfrak{A})$ is a balanced signed graph. If there is at least one edge with a $-$ sign in $OPS(\mathfrak{A})$, then $\mathfrak{A}$ is not an abelian group. However, in general, the converse of this statement is untrue. All of $OPS(\mathfrak{A})$'s edges for non-abelian groups of prime power order will have the sign '+.

Definition 3.29 (General order prime signed graph [70])**.** *The general order prime signed graph $GOPS(\mathfrak{A})$ of a finite group $\mathfrak{A}$ is the signed graph $((GOP(\mathfrak{A}), \sigma)$ where the function $\sigma : E(GOP(\mathfrak{A})) \to \{+, -\}$ is given by*

$$\sigma(\overline{(a,b)}) = \begin{cases} +, & \text{if } ab = ba; \\ -, & \text{otherwise.} \end{cases}$$

Let S be a non-empty set of positive integers and let $\mathfrak{A}$ be a finite group of order n. We define set-prime signed graph of $\mathfrak{A}$ and present some results.

Definition 3.30. *The set-prime graph $G_S(\mathfrak{A})$ of $\mathfrak{A}$ with respect to S is defined as a graph with vertex set $V(G_S(\mathfrak{A})) = \mathfrak{A}$ and any two distinct vertices a and b are adjacent in $G_S(\mathfrak{A})$ if and only if $(|a|, |b|) \in S$.*

Definition 3.31. *The set-prime signed graph $PG_S(\mathfrak{A})$ of $\mathfrak{A}$ is the signed graph $(G_S(\mathfrak{A}), \sigma)$, where the function $\sigma : E(G_S(\mathfrak{A})) \to \{+, -\}$ is given by*

$$\sigma(\overline{(a,b)}) = \begin{cases} +, & \text{if } ab = ba; \\ -, & \text{otherwise.} \end{cases}$$

The following are the some important results on Chapter 4:

Proposition 3.32. *$PG_{\{1\}}(\mathfrak{A}) \subsetneq PG_{\{1,q\}}(\mathfrak{A})$ if $\mathfrak{A}$ is a finite group of order n and $q \neq 2$ is a prime divisor of n.*

Proposition 3.33. *$PG_S(\mathfrak{A})$ is a connected balanced signed graph if $\mathfrak{A}$ is a finite abelian group of order n and S is a set containing 1.*

Proposition 3.34. *If $PG_S(\mathfrak{A})$ contains atleast one edge with $-$ sign, then the group $\mathfrak{A}$ is non-abelian.*

Proposition 3.35. *Let S_1 and S_2 be sets of positive integers and let $\mathfrak{A}$ be a finite group. If $S_1 \subseteq S_2$, then $PG_{S_1}(\mathfrak{A}) \subseteq PG_{S_2}(\mathfrak{A})$.*

Proposition 3.36. *Let $\mathfrak{A}$ be a group of order n and let S be a set of positive integers with $1 \in S$. Then $\{e\}$ is a dominating set in the underlying undirected graph of $PG_S(\mathfrak{A})$. Additionally, in the underlying undirected graph of $PG_S(\mathfrak{A})$ (i.e., in the set-prime graph $G_S(\mathfrak{A})$), $\{x\}$ is a dominating set if $x \in \mathfrak{A}$ and $|x|$ is a prime number with $|x| \in S$.*

Proposition 3.37. *The set-prime signed graph $PG_{\{1\}}(\mathfrak{A})$ for a group $\mathfrak{A}$ is a complete graph whose edges are assigned + sign if and only if $o(\mathfrak{A}) = 2$.*

Proposition 3.38. *For any group $\mathfrak{A}$ and $S = \{1\}$, the set-prime signed graph $PG_S(\mathfrak{A})$ can never be a unicyclic graph.*

Theorem 3.39. *If $\mathfrak{A}$ is a finite group of order $n = p^\alpha$ where p is a prime number and $\alpha \in \mathbb{Z}^+$, then $PG_{\{1\}}(\mathfrak{A}) \cong K^+_{1,n-1}$.*

Proposition 3.40. *Let $\mathfrak{A}$ be a finite cyclic group and $S \subseteq \mathbb{N}$. Then*

1. *$PG_S(\mathfrak{A})$ is a homogeneous signed graph with $+$ sign on all edges.*
2. *$PG_{\{1\}}(\mathfrak{A})$ has at least two pendent vertices.*
3. *$PG_{\{1,q\}}(\mathfrak{A})$ has at least two pendent vertices, whenever q is a prime not dividing $o(\mathfrak{A})$.*

Theorem 3.41. *If S is any non-empty set of positive integers and $\mathfrak{A}_1$, $\mathfrak{A}_2$ are two groups such that $\mathfrak{A}_1 \cong \mathfrak{A}_2$, then $PG_S(\mathfrak{A}_1) \cong PG_S(\mathfrak{A}_2)$.*

Theorem 3.42. *Let $\mathfrak{A}$ be a group and S be a set of positive integers. Then $Aut(\mathfrak{A}) \subseteq Aut(PG_S(\mathfrak{A}))$.*

Theorem 3.43. *Let $\mathfrak{A}$ be a finite group. Suppose that the signed graph $PG_{\{1\}}(\mathfrak{A})$ has two adjacent vertices a, b such that $|a||b| = o(\mathfrak{A})$. Then the set $\{a, b\}$ is a generating set of $\mathfrak{A}$. Moreover, if the edge (a, b) has '$+$' sign if and only if $\mathfrak{A}$ is abelian.*

Theorem 3.44. *Let $\mathfrak{A}$ be a finite abelian group. Let $X \subseteq \mathfrak{A}$ be such that the graph induced by X is complete (in the sense of unsigned(undirected) graph) in the signed graph $PG_{\{1\}}(\mathfrak{A})$ and the product of the order of all elements of X is same as of $o(\mathfrak{A})$. Then X is a generating set of $\mathfrak{A}$.*

Theorem 3.45. *Let $\mathfrak{A}$ be a finite group with $o(\mathfrak{A}) = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$, where p_i's are prime numbers and $n_i \in \mathbb{Z}^+$, $(1 \leq i \leq k)$. Then the set-prime signed graph $PG_{\{1\}}(\mathfrak{A})$ is a complete $(k+1)$-partite graph (in the sense of unsigned(undirected) graph) if and only if $|a| = p_i^j$, $\forall a \in \mathfrak{A} - \{e\}$, $1 \leq i \leq k$ and $1 \leq j \leq n_i$.*

Theorem 3.46. *Let $\mathfrak{A}$ be a finite group of order n, and let S be a set of positive integers containing 1 and all primes $< n$. Then, $PG_S(\mathfrak{A}) \cong K^+_{1,n-1}$ if and only if $o(\mathfrak{A}) = n$ is a prime.*

Definition 3.47. *The arithmetic function signed graph $SG_h(\mathfrak{A})$ of a finite group $\mathfrak{A}$ is the signed graph $(G_h(\mathfrak{A}), \sigma)$ where the function $\sigma : E(G_h(\mathfrak{A})) \to \{+, -\}$ is given by*

$$\sigma(\overline{(a,b)}) = \begin{cases} +, & \text{if } ab = ba; \\ -, & \text{otherwise.} \end{cases}$$

Proposition 3.48. *Negation of an arithmetic function signed graph of a finite group of order > 2 with respect to an arithmetic function h with $h(1) = 1$ will not be an arithmetic function signed graph.*

Proposition 3.49. *If $\mathfrak{A}$ is a group of order n, then for any arithmetic function h with $h(1) = 1$, $SG_h(\mathfrak{A})$ is a connected signed graph and the maximum positive degree $\Delta^+(SG_h(\mathfrak{A})) = n - 1$.*

Proposition 3.50. *If $\mathfrak{A}$ is abelian, then $SG_h(\mathfrak{A})$ is a homogeneous signed graph with positive sign on all edges (and hence it is a balanced signed graph).*

Proposition 3.51. *If $SG_h(\mathfrak{A})$ contains at least one edge with '$-$' sign (i.e., $SG_h(\mathfrak{A})$ is heterogeneous), then the group $\mathfrak{A}$ is non-abelian.*

Theorem 3.52. *If h is a multiplicative function, then for any non-abelian group of prime power order p^α, $SG_h(\mathfrak{A})$ is a homogeneous signed graph which is a star with '+' sign on all the edges (and hence it is a balanced signed graph).*

Theorem 3.53. *If h is a multiplicative function, then for any non-abelian group $\mathfrak{A}$ whose order is neither a prime nor a prime power, the signed graph $SG_h(\mathfrak{A})$ is heterogeneous and not balanced.*

Theorem 3.54. *If h is a completely multiplicative function, then for any finite non-abelian group $\mathfrak{A}$, the signed graph $SG_h(\mathfrak{A})$ is heterogeneous and not balanced.*

Definition 3.55. *Let f be an arithmetic function. A group $\mathfrak{A}$ is said to be f-balanced if the signed graph $SG_f(\mathfrak{A})$ is balanced; otherwise we say that $\mathfrak{A}$ is f-unbalanced.*

Theorem 3.56. *(i) Every finite abelian group is f-balanced for any arithmetic function f.*

(ii) A non-abelian group of prime power order is f-balanced for any multiplicative function f.

(iii) A non-abelian group whose order is not a prime power, is f-unbalanced for any multiplicative function f.

(iv) A finite non-abelian group is f-unbalanced for any completely multiplicative function f.

3.5 CHAPTER 5: SPECTRUM ENERGY OF ORDER PRIME SIGNED GRAPHS AND ARITHMETIC FUNCTION SIGNED GRAPHS

Let $\mathcal{S} = (G, \sigma)$ be a signed graph, where G is the underlying graph of $\mathcal{S}$ with vertex set $V(G) = \{u_1, u_2, \cdots, u_n\}$ and $\sigma : E(G) \to \{+, -\}$ is a signing function. The adjacency matrix of the signed graph $\mathcal{S}$, denoted by $A(\mathcal{S})$, is the square matrix (x_{ij}) where

$$x_{ij} = \begin{cases} 1, & \text{if } u_iu_j \in E(G) \text{ and } \sigma(u_iu_j) = +; \\ -1, & \text{if } u_iu_j \in E(G) \text{ and } \sigma(u_iu_j) = -; \\ 0, & \text{otherwise.} \end{cases}$$

An adjacency matrix completely describes a signed graph.

The eigenvalues of the adjacency matrix $A(\mathcal{S})$ of $\mathcal{S}$ are called the eigenvalues of $\mathcal{S}$. The set of eigenvalues of $\mathcal{S}$ is also called the spectrum of $\mathcal{S}$ and we denote this by $Spec(\mathcal{S})$.

If the distinct eigenvalues of $\mathcal{S}$ are $\gamma_1 > \gamma_2 > \cdots > \gamma_k$ and their multiplicities are $s_1, s_2, \ldots, s_k$, respectively, then the spectrum of $\mathcal{S}$ is

$$Spec(\mathcal{S}) = \begin{pmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_k \\ s_1 & s_2 & \cdots & s_k \end{pmatrix}.$$

The following are the some important results on Chapter 5:

Proposition 3.57. *Let f be an arithmetic function and $\mathfrak{A}$ be a finite group. If $\mathfrak{A}$ is f-balanced, then $f_{se}(\mathfrak{A}) = \mathcal{E}_f(\mathfrak{A})$.*

Proposition 3.58. *Let f be a completely multiplicative arithmetic function and $\mathfrak{A}$ be a finite group of order n. Then*

(i) If $\mathfrak{A}$ is abelian, then $f_{se}(\mathfrak{A}) = 2n - 2$.

(ii) If $\mathfrak{A}$ is non-abelian, then $f_{se}(\mathfrak{A}) = \mathcal{E}((K_n, \sigma))$, where σ is the signing function given in the Definition 3.47.

Theorem 3.59. *Let f be an arithmetic function and $\mathfrak{A}$ be a finite group. If f_s-energy of $\mathfrak{A}$ is a rational number, then it is an even integer.*

Theorem 3.60. *Let f be an arithmetic function and $\mathfrak{A}$ be a finite group. The f_s-energy of $\mathfrak{A}$ can never be a r-th root of a positive rational number which is not an integer.*

Theorem 3.61. *Let f be an arithmetic function and $\mathfrak{A}$ be a finite group. The f_s-energy of $\mathfrak{A}$ cannot be of the form $2^{\frac{s}{t}} a^{\frac{1}{t}}$, where a is odd, $t \geq 1$ and $0 \leq s \leq t-1$.*

Theorem 3.62. *If h is a multiplicative function, then for any group of prime power order p^η, where η is a positive integer.*

(i) $SG_h(\mathfrak{A}) = K^+_{1,p^\eta - 1}$.

(ii) $Spec(SG_h(\mathfrak{A})) = \begin{pmatrix} \sqrt{p^\eta - 1} & 0 & -\sqrt{p^\eta - 1} \\ 1 & p^\eta - 2 & 1 \end{pmatrix}$,

and hence

$$h_{se}(\mathfrak{A}) = 2\sqrt{p^\eta - 1}.$$

Theorem 3.63. *For a non-abelian finite group $\mathfrak{A}$ and a multiplicative function f, the following statements are equivalent:*

(i) $\mathfrak{A}$ is of prime power order.

(ii) $\mathfrak{A}$ is f-balanced.

(iii) $Spec(SG_f(\mathfrak{A})) = Spec(G_f(\mathfrak{A}))$.

Theorem 3.64. *For a non-abelian finite group $\mathfrak{A}$, the following statements are equivalent:*

(i) $\mathfrak{A}$ is of prime power order.

(ii) $OPS(\mathfrak{A})$ *is balanced.*

(iii) $Spec(OPS(\mathfrak{A})) = Spec(OP(\mathfrak{A}))$.

3.6 CHAPTER 6: CONCLUSION AND SCOPE FOR FUTURE WORK

Group theory has three historical foundations, namely, the theory of algebraic equations, number theory and geometry. The concept of arithmetic function graphs of finite groups is a topic that connects graph theory, group theory and number theory.

The further study of arithmetic function graphs (undirected/signed) of finite groups generates a good number of research problems in the areas of graph (undirected/signed) theory, group theory and number theory. Some of them are listed below in the form of future scope:

Graph (undirected/signed) theory:

1. Studying properties of various arithmetic functions through corresponding arithmetic function graphs of finite groups.
2. Setting up new relations between partition function and its arithmetic function graphs of finite groups.

Number theory:

1. Characterizing graphs which are arithmetic function graphs.
2. Characterizing signed graphs which are arithmetic function signed graphs.

Group theory:

1. Finding properties of groups using arithmetic function graphs and arithmetic function signed graphs.
2. Classification of groups through various arithmetic functions via arithmetic function graphs and arithmetic function signed graphs.

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4 List of Publications/Papers Presented/ Communicated

4.1 Publications in Journals/ Communicated

1. Arithmetic Function Graph of a Finite Group, *Palestine Journal of Mathematics*, 11(2) (2022), 488-495. (Scopus Indexed) **(ISSN: 2219-5688, Hebron, Palestine)** (With R. Rajendra and P. S. K. Reddy)

2. Eigenvalues and Energy of Arithmetic Function Graph of a Finite Group, *Proceedings of the Jangjeon Mathematical Society*, 27(1) (2024), *to appear.* (Scopus Indexed) **(ISBN: 89-87809-15-3, ISSN:1598-7264, South Korea)** (With R. Rajendra and P. S. K. Reddy)

3. Order prime signed graphs and arithmetic function signed graphs, Communicated for Publication. (With R. Rajendra and P. S. K. Reddy)

4. Spectrum energy of order prime signed graphs and arithmetic function signed graphs, Communicated for Publication. (With R. Rajendra and P. S. K. Reddy)

4.2 Papers presented in an International/National conference

1. Presented a Paper titled "On Energy of General Order Prime Graphs of Finite Group" in the National Conference on "Graph Theory and Discrete Mathematics" organized by Department of Mathematics, Mangalore University, Mangaluru during 05-06 February 2018.

2. Presented a paper titled "On Order Prime Signed Graph of a Finite Group" in the One Day International Conference on Recent Advances in Interdisciplinary Mathematical Sciences (RAIMS-2019) held at Government Science College, Bengaluru on 6th September, 2019.

3. Presented a Paper titled "Set-Prime Signed Graphs of Finite Groups" in the Virtual International Conference on Discrete Mathematics-2020 (VICDM-2020) jointly organized by Department of Mathematics, Mangalore University, Mangaluru and Academy of Discrete Mathematics and Applications (ADMA) during 08-10 February 2021.

4. Presented a paper titled "Arithmetic Function Graph of a Finite Group" in the International Conference on "Special Functions & Applications (ICSFA-2022)" organized by DOS in Mathematics, University of Mysore, Mysuru during November 26-28, 2022.

4.3 Workshops/Conference attended

1. Participated in the One week online Faculty development Programme on "Mathematical and Statistical Modelling" organized by Godavari Institute of Engineering and Technology (Autonomous) during 26-30 May 2020.

2. Participated in the One week online Faculty development Programme on "Recent Advances in Material Science and Mathematical Application" organized by Department of Basic Science, Ghousia College of Engineering, Ramanagara during 15-19 June 2020.

3. Participated in the one day webinar on "Advances in Graph Theory" organized by the Ramanujan Club, Department of Mathematics, Government College Daman on 29-06-2020.

4. Participated in the one day webinar on "Abstract Algebra and its Applications" organized by Department of Mathematics, A. E. S National Degree College, Gauribidanur on 30-06-2020.

5. Participated in the virtual Workshop on "Domination Theory and Topological Indices" jointly organized by Department of Mathematics, Mangalore University, Mangaluru and Academy of Discrete Mathematics and Applications (ADMA) during 04-06 February 2021.

Signature of the Candidate
(Kemparaju R.)

Signature of the Guide
(Dr. Rajendra R)